

GCD SUMS FROM POISSON INTEGRALS AND SYSTEMS OF DILATED FUNCTIONS

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ABSTRACT. Upper bounds for GCD sums of the form

$$\sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}$$

are proved, where $(n_k)_{1 \leq k \leq N}$ is any sequence of distinct positive integers and $0 < \alpha \leq 1$; the estimate for $\alpha = 1/2$ solves in particular a problem of Dyer and Harman from 1986, and the estimates are optimal except possibly for $\alpha = 1/2$. The method of proof is based on identifying the sum as a certain Poisson integral on a polydisc; as a byproduct, estimates for the largest eigenvalues of the associated GCD matrices are also found. The bounds for the GCD sums are used to settle two longstanding problems on the a.e. behavior of systems of dilated functions: the a.e. growth of sums of the form $\sum_{k=1}^N f(n_k x)$ and the a.e. convergence of $\sum_k c_k f(kx)$ when f is 1-periodic and of bounded variation or in $\text{Lip}_{1/2}$.

1. INTRODUCTION

This paper studies two closely related topics: Greatest common divisor (GCD) sums of the form

$$(1) \quad \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}$$

for $0 < \alpha \leq 1$ and convergence properties of systems of dilated functions $f(n_k x)$ on the unit interval $[0, 1]$. Here (n_k) is a sequence of distinct positive integers and f is a 1-periodic real-valued function of bounded variation or belonging to the class $\text{Lip}_{1/2}$. We will introduce a new method for estimating sums of the form (1) and in particular solve a problem posed by Dyer and Harman in [12]. In addition, using estimates for (1), we will settle two longstanding problems regarding the a.e. behavior of systems of dilated functions.

Date: October 22, 2012.

2010 *Mathematics Subject Classification.* 11C20, 42A20, 42A61, 42B05.

The first author is supported by a Schrödinger scholarship of the Austrian Research Foundation (FWF). The second author is supported by the Research Council of Norway grant 185359/V30. This paper was written while the second author participated in the research program *Operator Related Function Theory and Time-Frequency Analysis* at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during 2012–2013.

The study of GCD sums like (1) was initiated by Erdős who around 1940 posed the problem of giving upper bounds when $\alpha = 1$, after observing that such sums can be used to estimate integrals of the form

$$\int_0^1 \left(\sum_{k=1}^N \mathbb{1}_{[a,b)}(\{n_k x\}) \right)^2 dx$$

(we write $\{\cdot\}$ for the fractional part). The problem was solved by Gál [13], who proved that¹

$$(2) \quad \frac{1}{N} \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^2}{n_k n_\ell} \leq c(\log \log N)^2,$$

and moreover that this bound is optimal up to the value of the absolute constant c . In 1986, Dyer and Harman [12] proved that

$$(3) \quad \frac{1}{N} \sum_{k,\ell=1}^N \frac{\gcd(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \leq C \exp \left(\frac{c \log N}{\log \log N} \right)$$

for two absolute constants C and c , and they used this estimate to prove results in metric Diophantine approximation; Dyer and Harman found also that

$$\frac{1}{N} \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \leq c(\alpha) \exp((\log N)^{(4-4\alpha)/(3-2\alpha)})$$

for $1/2 < \alpha < 1$. In his monograph [16], Harman writes that “it is tempting to conjecture” that the right-hand side of (3) can be replaced by a constant times $\exp(c\sqrt{\log N}/\log \log N)$. One of our examples given below will disprove this conjecture and show that here we can not have a function smaller than $\exp(2\sqrt{(\log N)/\log \log N})$. However, the following theorem, which is our main result on GCD sums, will “almost” confirm Harman’s conjecture, and it gives optimal upper bounds for (1) when $1/2 < \alpha < 1$.

Theorem 1. *For arbitrary N -tuples of distinct positive integers n_1, n_2, \dots, n_N , we have*

$$\frac{1}{N} \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \leq \begin{cases} \exp((c(\alpha) + o(1))(\log N)^{1-\alpha}(\log \log N)^{-\alpha}), & 1/2 < \alpha < 1 \\ \exp(4(\log 2)^{-1/2} + o(1))(\log N \log \log N)^{1/2}), & \alpha = 1/2 \\ N^{(1+o(1))(1-2\alpha)}, & 0 < \alpha < 1/2 \end{cases}$$

when $N \rightarrow \infty$, where

$$c(\alpha) = \frac{c}{(1-\alpha)\sqrt{2\alpha-1}}$$

for an absolute constant c ; the induced constants in the $o(1)$ terms depend only on α .

¹Here and in what follows we may assume that $N \geq 3$ so that $\log \log N$ is well defined and positive.

We will in fact prove a more general result than Theorem 1 which can be given a function theoretic interpretation on the infinite-dimensional polydisc \mathbb{D}^∞ . Indeed, the observation underlying our proof is that the GCD sum (1) can be written as a certain Poisson integral.

We will show by an example that Theorem 1 is best possible (up to a constant factor in the exponent) when $1/2 < \alpha < 1$. We will also see that the blow-up of the constant in the exponent is of the right magnitude when $\alpha \nearrow 1$. We conjecture that the blow-up of the constant $c(\alpha)$ when $\alpha \searrow 1/2$ is an artifact and that the estimate in the range $1/2 < \alpha < 1$ should indeed extend to $\alpha = 1/2$, which would then be optimal too. On the other hand, as we will see, the estimates change abruptly when we pass from $\alpha = 1/2$ to $\alpha < 1/2$, as a consequence of the divergence of the series $\sum p^{-2\alpha}$ over the prime numbers p ; the slow divergence when $\alpha = 1/2$ is the reason why this is a particularly delicate case. The range $0 < \alpha < 1/2$, included here for the sake of completeness, is less subtle, and it is easy to give an example showing that the estimate of Theorem 1 is best possible.

The proof of Theorem 1 as well as these examples will be presented in Section 3 below. An immediate consequence of our reformulation in terms of Poisson integrals is that the corresponding matrices are positive definite. In the subsequent Section 4 we will see that in turn Theorem 1 implies precise estimates for the largest eigenvalues of these matrices, or, equivalently, for their spectral norms.

2. APPLICATIONS

Our first application of Theorem 1 concerns the asymptotic growth of $\sum_{k=1}^N f(n_k x)$, where $(n_k)_{k \geq 1}$ is a sequence of distinct integers and $f \in \text{BV}$ is a function satisfying

$$(4) \quad \int_0^1 f(x) dx = 0, \quad f(x+1) = f(x)$$

(here, and in the sequel, we write $f \in \text{BV}$ if $\text{Var}_{[0,1]} f < \infty$). Applying a result of Koksma [21], Gál used (2) to prove that

$$(5) \quad \left| \sum_{k=1}^N f(n_k x) \right| = \mathcal{O} \left(\sqrt{N} (\log N)^{3/2+\varepsilon} \right) \quad \text{a.e.}$$

Alternatively, (5) can also be obtained without the use of Gál's theorem by a combination of the Carleson–Hunt inequality, the Erdős–Túran inequality, and Koksma's inequality (see [4]). Recently, Aistleitner, Mayer, and Ziegler [2] used (3) to show that (5) can be improved to

$$\left| \sum_{k=1}^N f(n_k x) \right| = \mathcal{O} \left(\sqrt{N} (\log N)^{3/2} (\log \log N)^{-1/2+\varepsilon} \right) \quad \text{a.e.}$$

On the other hand, Berkes and Philipp [6] constructed an increasing sequence of integers $(n_k)_{k \geq 1}$ for which

$$(6) \quad \limsup_{N \rightarrow \infty} \frac{\left| \sum_{k=1}^N \cos 2\pi n_k x \right|}{\sqrt{N \log N}} = \infty \quad \text{a.e.,}$$

which implies that the exponent of the logarithmic factor in (5) in general cannot be reduced from $3/2 + \varepsilon$ to $1/2$. Our next theorem shows that this exponent can be taken as $1/2 + \varepsilon$ for arbitrary $\varepsilon > 0$, which is then best possible up to a factor smaller than any positive power of $\log N$. Here and in what follows, Lip_α denotes the class of functions f that are Hölder continuous with exponent α .

Theorem 2. *Let $(n_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers, let f be a function satisfying (4) and assume in addition that either $f \in \text{BV}$ or $f \in \text{Lip}_{1/2}$. Then for any $\varepsilon > 0$*

$$\left| \sum_{k=1}^N f(n_k x) \right| = \mathcal{O} \left(\sqrt{N} (\log N)^{1/2+\varepsilon} \right) \quad \text{a.e.}$$

when $N \rightarrow \infty$.

A closely related problem asks similarly for the a.e. asymptotic order of the discrepancy D_N of $(\{n_k x\})_{k \geq 1}$. (See [11, 22] for general information on discrepancy theory.) Baker [4] proved that

$$(7) \quad D_N(\{n_1 x\}, \dots, \{n_N x\}) = \mathcal{O} \left(N^{-1/2} (\log N)^{3/2+\varepsilon} \right) \quad \text{a.e.},$$

and by (6) and Koksma's inequality the exponent of the logarithmic term in (7) can not be reduced below $1/2$. Unfortunately, our method does not apply to the problem of finding an optimal upper bound for the discrepancy, in which case one needs to consider the supremum for every x over a family of indicator functions.

Our second application of Theorem 1 deals with the a.e. convergence of series of the form

$$(8) \quad \sum_{k=1}^{\infty} c_k f(kx)$$

for 1-periodic functions f . The cases $f(x) = \cos 2\pi x$ and $f(x) = \sin 2\pi x$ were settled by Carleson in his celebrated work [9], which shows that

$$\sum_{k=1}^{\infty} c_k \cos 2\pi kx \quad \text{and} \quad \sum_{k=1}^{\infty} c_k \sin 2\pi kx$$

are a.e. convergent whenever

$$(9) \quad \sum_{k=1}^{\infty} c_k^2 < \infty.$$

Carleson's theorem is a deep and difficult result with several astonishing consequences (see e.g. Lemma 3 below); for a comprehensive presentation of Carleson's theorem, we refer to the monographs [3, 26] and to the survey article [23].

Gapoškin [14] observed that as a direct consequence of Carleson's theorem, (9) also implies the a.e. convergence of (8) for functions f satisfying (4), under the additional assumption that f is in the Wiener algebra. This containment is guaranteed by the condition that $f \in \text{Lip}_\alpha$ for some α in $(1/2, 1]$. On the other hand, a result of Berkes [5] shows

that (9) fails to be sufficient to yield the a.e. convergence of (8) whenever (4) holds and $f \in \text{Lip}_{1/2}$. In the positive direction, Gapoškin [15] proved that (8) is a.e. convergent for a function f satisfying (4) and $f \in \text{Lip}_{1/2}$, provided

$$(10) \quad \sum_{k=1}^{\infty} c_k^2 (\log k)^\mu < \infty$$

for some $\mu > 3$. Berkes and Weber [7] showed that it is sufficient to assume $\mu > 2$. Our Theorem 3 below shows that it is sufficient to assume (10) for any positive μ . This result is optimal, up to factors smaller than any positive power of $\log k$.

Theorem 3. *Let f be a function satisfying (4), and assume in addition that either $f \in \text{BV}$ or $f \in \text{Lip}_{1/2}$. Then the sum*

$$\sum_{k=1}^{\infty} c_k f(kx)$$

is convergent for almost all x whenever

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^\varepsilon < \infty$$

for some $\varepsilon > 0$.

There are many other applications of Theorem 1. For example, in [1] it is proved that (8) is a.e. convergent for f satisfying (4) and $f \in \text{Lip}_\alpha$ for some $\alpha \in [1/4, 1/2)$, provided

$$(11) \quad \sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{2 \log k}{\log \log k}\right) < \infty.$$

Using Theorem 1 instead of (3), we see that condition (11) can be relaxed to

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(3(\log k \log \log k)^{1/2}\right) < \infty,$$

cf. recent related results of Brémont [8] and Weber [27].

A further application of Theorem 1 yields an improvement of a result of Harman [18] on metric Diophantine approximation. The original proof uses the estimate (3), which can be replaced by our Theorem 1 in order to improve a factor of order $\exp(c \log / \log \log N)$ to a factor of order $\exp(c \sqrt{\log N \log \log N})$. This result is connected with the Duffin–Schaeffer conjecture, a notoriously difficult open problem from metric Diophantine approximation (see [16, 17]).

3. PROOF OF THEOREM 1 VIA TRIGONOMETRIC POLYNOMIALS ON \mathbb{D}^∞

We begin by fixing some notation. We set

$$\Gamma_{p^{-\alpha}}(N) = \sup_{n_1, \dots, n_N} \frac{1}{N} \sum_{k, \ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha},$$

where the supremum is taken over all distinct positive integers n_1, \dots, n_N . Here the index $p^{-\alpha}$ hints at the role played by the prime numbers; this notation will appear natural once we have transformed the problem of estimating $\Gamma_{p^{-\alpha}}$ into a problem on the infinite-dimensional polydisc \mathbb{D}^∞ .

We next introduce multi-index notation suitable for our purposes. A multi-index is a sequence $\beta = (\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(R)}, 0, 0, \dots)$ consisting of nonnegative integers with only a finite number of them being nonzero. We let $\text{supp } \beta$ be the finite set of positive integers j for which $\beta^{(j)} > 0$; we write $R(\beta)$ for the maximal element in $\text{supp } \beta$. Two multi-indices β and μ may be added and subtracted as sequences. Then $\beta - \mu$ may fail to be a multi-index, but the sequence $|\beta - \mu| = (|\beta^{(j)} - \mu^{(j)}|)$ will again be a multi-index. We may multiply multi-indices by positive integers in the obvious way and express any multi-index as a linear combination of the natural basis elements e_j , where e_j is the multi-index supported by $\{j\}$ with $e_j^{(j)} = 1$. We write $\beta \leq \mu$ if $\beta^{(j)} \leq \mu^{(j)}$ for every j . For a sequence of complex numbers $z = (z_j)$, we use the notation

$$z^\beta = \prod_{j=1}^{R(\beta)} z_j^{\beta^{(j)}};$$

we will sometimes write $z^{-\beta}$ for the number $(z^\beta)^{-1}$.

We write $p = (p_j)$ for the sequence of prime numbers ordered by increasing magnitude. If $n_k = p^{\beta_k}$, then we may write

$$\frac{(\gcd(n_k, n_\ell))^2}{n_k n_\ell} = p^{-|\beta_k - \beta_\ell|}.$$

For an arbitrary sequence t of positive numbers in \mathbb{D}^∞ and a set of distinct multi-indices $B = \{\beta_1, \dots, \beta_N\}$, we now define

$$S(t, B) = \frac{1}{N} \sum_{k, \ell=1}^N t^{|\beta_k - \beta_\ell|}.$$

We set

$$\Gamma_t(N) = \sup_B S(t, B),$$

where the supremum is taken over all possible sets B of distinct multi-indices β_1, \dots, β_N . Note that now we only need to declare that $p^{-\alpha} = (p_j^{-\alpha})$ to make our notation consistent.

For a minor technical reason, we introduce the following notation. For numbers $0 < a < 1$, we define

$$2 \star a = \begin{cases} 2a, & 0 < a < 1/2 \\ a, & 1/2 \leq a < 1, \end{cases}$$

and for a sequence t we set $2 \star t = (2 \star t_j)$. For a decreasing sequence t of positive numbers in the sequence space c_0 , we define

$$\kappa(t) = \begin{cases} 0 & \text{if } t_1 < 1/2 \\ \max\{j : t_j \geq 1/2\} & \text{otherwise.} \end{cases}$$

We will prove the following general theorem.

Theorem 4. *Let $t = (t_j)$ be a sequence of positive numbers in $\mathbb{D}^\infty \cap c_0$ such that $2 \star t = (\tau_j)$ is a decreasing sequence. Fix a positive number $\xi > (\log 2)^{-1}$, and set $r_N = [\xi \log N] + \kappa(t)$. Then, for arbitrary numbers $1 > v_1 \geq v_2 \geq \dots \geq v_{r_N}$ satisfying also $v_j > \tau_j^2$ for $1 \leq j \leq r_N$, we have*

$$\Gamma_t(N) \leq \prod_{j=1}^{r_N} (1 - v_j)^{-1} (1 - v_j^{-1} \tau_j^2)^{-1} \prod_{k=r_N+1}^{N-1} (1 - v_{r(N)}^{-1} \tau_k^2)^{-1} + \exp \left(C \sum_{\ell=1}^{N-1} t_\ell^2 \right),$$

where C is a positive constant depending only on ξ .

This theorem is clearly applicable when the sequence t is in ℓ^2 , but it can also be used when the series $\sum_j t_j^2$ is “slowly” divergent, as we will now see.

Proof of Theorem 1. We now take Theorem 4 for granted and show that it implies Theorem 1. When $1/2 < \alpha < 1$, we choose $v_j = \max(\tau_j, (2\alpha - 1)^{-1/2} \tau_{r_N})$ with $\tau_j = 2 \star p_j^{-\alpha}$. (The decay of τ_j is a minor technical point which can be dealt with by an obvious rearrangement of the sequence.) The verification of the estimates in Theorem 1 is then a straightforward matter if we use the prime number theorem to estimate the products appearing in Theorem 4. For $\alpha = 1/2$, we choose $v_j = \max(2 \star p_j^{-1/2}, (2 \log 2 \log \log N)^{1/2} / (\log N)^{1/2})$ and use similarly the prime number theorem to estimate the products.

Finally, to deal with the case $0 < \alpha < 1/2$, we apply Hölder’s inequality with exponents $1/(2\alpha)$ and $1/(1 - 2\alpha)$:

$$\frac{1}{N} \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \leq \left(\sum_{k,\ell=1}^N \frac{\gcd(n_k, n_\ell)}{(n_k n_\ell)^{1/2}} \right)^{2\alpha} N^{1-4\alpha}.$$

Taking into account the definition of $\Gamma_{p^{-1/2}}(N)$, we therefore get

$$\frac{1}{N} \sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \leq (\Gamma_{p^{-1/2}}(N))^{2\alpha} N^{1-2\alpha},$$

which, in view of what we already proved regarding $\Gamma_{p^{-1/2}}(N)$, gives the desired result. \blacksquare

To see to what extent the result is sharp for $1/2 \leq \alpha < 1$, we consider the following example: Set $N = 2^r$ and take n_1, \dots, n_N to be all square-free numbers composed of the first r primes. Then

$$\sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} = N \prod_{j=1}^r (1 + p_j^{-\alpha}),$$

which follows from an argument in [13, p. 21]. By the prime number theorem, we therefore get

$$\sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} = N \exp(((1-\alpha)^{-1} + o(1))(\log N)^{1-\alpha}(\log \log N)^{-\alpha}).$$

Thus our estimate is of the right order of magnitude when $1/2 < \alpha < 1$, as is the blow-up of the constant $c(\alpha)$ when $\alpha \nearrow 1$. However, this example does not settle the cases $\alpha \searrow 1/2$ and $\alpha = 1/2$. In fact, we see that there is a discrepancy of a factor $\log \log N$ in the exponent between our estimate and the lower bound obtained from the example. It seems likely that the blow-up of the constant $c(\alpha)$ when $\alpha \searrow 1/2$ is an artifact. The trouble is that the divergence of the series $\sum_j p_j^{-1}$ implies that the number of primes involved in the sum plays a role. We believe the number of primes should be $O(\log N)$ when the sum is maximal, but can only infer from our method of proof that this number is bounded by $N - 1$.

Our estimate is however optimal when $0 < \alpha < 1/2$, up to a possible improvement of the $o(1)$ term. To see this, it suffices to consider the example $n_1 = 2, n_2 = 3, \dots, n_N = p_N$. It is then immediate from the prime number theorem that

$$\sum_{k,\ell=1}^N \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} = N^{1-2\alpha+o(1)} N.$$

The reason for the abrupt change at $\alpha = 1/2$ is that the relatively fast divergence of $\sum_j p_j^{-2\alpha}$ (as in this example) plays a dominant role when $0 < \alpha < 1/2$.

We will now prepare for the proof of Theorem 4 by making the passage to Poisson integrals as alluded to above. We let σ_K denote normalized Lebesgue measure on the unit polycircle \mathbb{T}^K and write

$$P_K(\zeta, z) = \prod_{k=1}^K \frac{1 - |\zeta_k|^2}{|1 - \bar{\zeta}_k z_k|^2},$$

which is the Poisson kernel for the unit polydisc \mathbb{D}^K at the point ζ . It is convenient in this definition to allow ζ to be a point in the infinite-dimensional polydisc \mathbb{D}^∞ . The only property of P_K needed is the identity

$$t^{|\beta-\mu|} = \int_{\mathbb{T}^K} z^\beta \bar{z}^\mu P_K(t, z) d\sigma_K(z),$$

valid for positive sequences t in \mathbb{D}^∞ , which is obtained by computing the integral over \mathbb{T}^K as an iterated integral over K copies of the unit circle. It leads immediately to the following lemma.

Lemma 1. *For a positive sequence t in \mathbb{D}^∞ , arbitrary multi-indices β_1, \dots, β_N with $K = \max_j R(\beta_j)$, and complex numbers c_1, \dots, c_N , we have*

$$(12) \quad \sum_{k,\ell=1}^N t^{|\beta_k - \beta_\ell|} c_k \bar{c}_\ell = \int_{\mathbb{T}^K} \left| \sum_{j=1}^N c_j z^{\beta_j} \right|^2 P_K(t, z) d\sigma_K(z).$$

The fact that the quadratic form on the left-hand side of (12) can be written as the square of a norm was first observed in [24] in the special case when $t = (p_j^{-\alpha})$ and $\alpha > 1/2$, based on ideas from [19]. The present formulation seems more illuminating and leads to an interesting problem for trigonometric polynomials on \mathbb{D}^∞ . We will take a closer look at this problem in the next section, where we will estimate the ℓ^2 -norm of the quadratic form on the left-hand side of (12), or, in other words, the largest eigenvalue of the matrix $(t^{|\beta_k - \beta_\ell|})$.

For the proof of Theorem 4, we only need (12) when $c_k \equiv 1$. Incidentally, this restriction is crucial for the combinatorial argument that leads to Lemma 2 below, which is our next auxiliary result. It is interesting to note that this lemma relies on the left-hand side of (12), while the subsequent analytic part of the proof of Theorem 4 departs from the right-hand side of this identity.

We will use a variant of Gál's terminology: A set B of N multi-indices β_1, \dots, β_N is said to be κ -canonical for $0 \leq \kappa < N$ if $\beta \in B$ and $e_j \leq \beta$ for $\kappa < j \leq N$ imply that $\beta - e_j \in B$. The following lemma is a modification of a theorem in [13, p. 17].

Lemma 2. *Suppose B is a set of N multi-indices. Let t be a decreasing sequence of positive numbers in $\mathbb{D}^\infty \cap c_0$. If $\kappa(t) < N$, then there exists a $\kappa(t)$ -canonical set of N multi-indices $B' = \{\beta'_1, \dots, \beta'_N\}$ such that $S(2 \star t, B') \geq S(t, B)$ and $\#\bigcup_{j=1}^N \text{supp } \beta'_j \leq N - 1$.*

Proof. We will modify B and t by an inductive algorithm. We break the argument into two parts, the first of which will give a set of multi-indices for which the union of their supports has cardinality at most $N - 1$.

Part 1: It will be convenient to use the following terminology. We say that a multi-index β in B is j -maximal if j is in $\text{supp } \beta$ but $(\beta^{(j)} + 1)e_j \not\leq \mu$ for every μ in B . We will construct from B a new set \tilde{B} with the property that if β in \tilde{B} is j -maximal, then also $\beta - e_j$ is in \tilde{B} , while at the same time $S(t, \tilde{B}) \geq S(t, B)$. Writing $\tilde{B} = \{\tilde{\beta}_1, \dots, \tilde{\beta}_N\}$, we see that, as a consequence, we will have $\#\bigcup_{j=1}^N \text{supp } \tilde{\beta}_j \leq N - 1$.

Fix a positive integer j in $\bigcup_k \text{supp } \beta_k$. Let ν be the largest integer such that $\nu e_j \leq \beta$ for some β in B . Suppose there is a j -maximal multi-index β in B such that $\nu e_j \leq \beta$ but $\beta - e_j$ is not in B . For every such β , we replace β in B by $\beta - e_j$; we call the new set of multi-indices B_ν . A term by term comparison shows that $S(t, B_\nu) \geq S(t, B)$.

If there is a j -maximal multi-index in B_ν with $\beta^{(j)} = \mu$, then it must have the desired property that also $\beta - e_j$ is in B_ν , and no further action is needed. In the opposite case, we repeat the argument with ν replaced by $\nu - 1$. The iteration terminates when either the desired property holds for some B_η with $1 \leq \eta \leq \nu$ or j is not in the support of any multi-index in B_1 .

We repeat this iteration for every j in $\bigcup_k \text{supp } \beta_k$ and obtain thus the desired set \tilde{B} .

Part 2: By part 1, we may from now on assume that, for every j in $\bigcup_k \text{supp } \beta_k$, any j -maximal multi-index β in B has the property that $\beta - e_j$ is in B . This is irrelevant for the argument to be given below, but we need it to reach the desired conclusion about the cardinality of $\bigcup_j \text{supp } \beta_j$.

We now assume that $\kappa(t) < N$. We fix a $j > \kappa(t)$ in $\bigcup_j \text{supp } \beta_j$ and divide B into disjoint subsets b_1, \dots, b_ℓ ($1 \leq \ell \leq N$), which we call j -chains of multi-indices, according to the following rule: two distinct multi-indices β and μ belong to the same j -chain b if $|\beta - \mu| = \eta e_j$ for some $\eta > 0$. This means that every element β in b is of the form $\beta = \mu + \eta e_j$, where $\mu^{(j)} = 0$ and μ is thus a multi-index that characterizes the j -chain b . We now modify each j -chain b_k by replacing it by the set

$$\tilde{b}_k = \{\mu, \mu + e_j, \dots, \mu + (\#b - 1)e_j\},$$

and we set $\tilde{B} = \bigcup_{k=1}^\ell \tilde{b}_k$.

It is immediate that $S(t, \tilde{b}) \geq S(t, b)$. To compare the terms of the sum corresponding to pairs of multi-indices from different j -chains, we introduce the notation

$$S(t; a, b) = \sum_{\beta \in a, \mu \in b} t^{|\beta - \mu|},$$

where a and b are two different j -chains. A simple combinatorial argument shows that

$$S(t; a, b) \leq \sum_{\beta \in \tilde{a}, \mu \in \tilde{b}, \beta^{(j)} = \mu^{(j)}} t^{|\beta - \mu|} + 2 \sum_{\beta \in \tilde{a}, \mu \in \tilde{b}, \beta^{(j)} \neq \mu^{(j)}} t^{|\beta - \mu|}.$$

This implies that $S(t; a, b) \leq S(t + t_j e_j; a, b)$ and, more generally, that $S(t + t_j e_j, \tilde{B}) \geq S(t, B)$.

The result follows if we make this modification in turn for every j in $\bigcup_k \text{supp } \beta_k$ for which $j > \kappa(t)$. ■

Proof of Theorem 4. By Lemma 2, it suffices to estimate $S(2 \star t, B)$ for every $\kappa(t)$ -canonical set $B = \{\beta_1, \dots, \beta_N\}$ of N multi-indices satisfying

$$\# \bigcup_{j=1}^N \text{supp } \beta_j \leq N - 1.$$

It is clear that we may assume that

$$\bigcup_{j=1}^N \text{supp } \beta_j = \{1, 2, \dots, K\}$$

for some $K \leq N - 1$ since we are seeking an upper bound for all sums $S(2 \star t, B)$ and $2 \star t$ is a decreasing sequence. Note that we may write

$$P_K(2 \star t, z) = \prod_{k=1}^K (1 - \tau_k^2) \left| \sum_{\beta: R(\beta) \leq K} (2 \star t)^\beta z^\beta \right|^2.$$

By Lemma 1 and the orthonormality of the monomials z^β , we therefore get

$$S(2 \star t, B) \leq \frac{1}{N} \sum_{\beta: R(\beta) \leq K} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2.$$

Let \mathcal{B}_1 denote the set of those multi-indices β such that $R(\beta) \leq K$ and $\#\text{supp } \beta \leq r_N$, and let \mathcal{B}_2 denote the set of all other multi-indices β with $R(\beta) \leq K$. By the Cauchy–Schwarz inequality, we get

$$\sum_{\beta \in \mathcal{B}_2} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2 \leq \sum_{\beta \in \mathcal{B}_2} N \sum_{j: \beta_j \leq \beta} (2 \star t)^{2(\beta - \beta_j)},$$

which may be written as

$$\sum_{\beta \in \mathcal{B}_2} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2 = \sum_{j=1}^N \sum_{\beta \in \mathcal{B}_2: \beta_j \leq \beta} N (2 \star t)^{2(\beta - \beta_j)}.$$

Since B is assumed to be $\kappa(t)$ -canonical, $\#\text{supp } \beta_j \leq (\log N)/\log 2 + \kappa(t)$ for every j , and hence $\#\text{supp}(\beta - \beta_j) \geq \varepsilon \log N$ for a positive ε , depending on our choice of ξ , when β is in \mathcal{B}_2 . We assume for convenience that $\varepsilon \log N$ is an integer. Suppose $2\tau_j^2 > e^{-1/\varepsilon}$ for $j = 1, \dots, J \leq N-1$. Then we may estimate the inner sum as an Euler product and obtain

$$\sum_{\beta \in \mathcal{B}_2} N (2 \star t)^{2(\beta - \beta_j)} \leq e^{J/\varepsilon} \prod_{j=1}^J (1 - \tau_j^2)^{-1} \prod_{k=J}^{N-1} (1 - \tau_k^2 e^{1/\varepsilon})^{-1},$$

which means that

$$\sum_{\beta \in \mathcal{B}_2} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2 \leq N \exp \left(C \sum_{j=1}^{N-1} t_j^2 \right)$$

for a constant C that only depends on ε .

We next consider the summation over \mathcal{B}_1 . Let β be an arbitrary multi-index in this set with

$$\text{supp } \beta = \{j_1, \dots, j_i\},$$

where $i \leq r_N$ by the definition of \mathcal{B}_1 . For any numbers v_k satisfying the hypothesis of Theorem 4, we define a sequence w_β by requiring

$$w_\beta^{(j_k)} = \begin{cases} v_k & \text{for } k = 1, \dots, i \\ 0 & \text{otherwise.} \end{cases}$$

We now apply the Cauchy–Schwarz inequality and get

$$\begin{aligned} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2 &\leq \sum_{j: \beta_j \leq \beta} w_\beta^{\beta - \beta_j} \sum_{k: \beta_k \leq \beta} w_\beta^{-(\beta - \beta_k)} (2 \star t)^{2(\beta - \beta_k)} \\ &\leq \prod_{j=1}^{r_N} (1 - v_j)^{-1} \sum_{k: \beta_k \leq \beta} w_\beta^{-(\beta - \beta_k)} (2 \star t)^{2(\beta - \beta_k)}, \end{aligned}$$

so that we may finally conclude that

$$\sum_{\beta \in \mathcal{B}_1} \left(\sum_{j: \beta_j \leq \beta} (2 \star t)^{\beta - \beta_j} \right)^2 \leq N \prod_{j=1}^{r_N} (1 - v_j)^{-1} (1 - v_j^{-1} \tau_j^2)^{-1} \prod_{k=r_N+1}^{N-1} (1 - v_{r(N)}^{-1} \tau_k^2)^{-1}.$$

■

It is worth pointing out that the most essential use of Lemma 2 was to reduce the problem to the case when the cardinalities $\#\text{supp } \beta_j$ are uniformly bounded by a constant times $\log N$. It would be desirable to find a way to arrive at this reduction without involving the auxiliary sequence $2 \star t$. In particular, if this could be done, then our method of proof would allow us to recapture Gál's theorem (2). Unfortunately, we may only conclude from Theorem 4 that $\Gamma_{p^{-1}}(N) = \mathcal{O}((\log \log N)^4)$ when $N \rightarrow \infty$.

4. SPECTRAL NORMS OF GENERALIZED GCD MATRICES

This section will show that we with little extra effort may obtain from Theorem 4 precise estimates for the largest eigenvalues of the matrices $(t^{|\beta_k - \beta_\ell|})$, which we will refer to as generalized GCD matrices. Since, by (12), these matrices are positive definite, we see that

$$\Lambda_t(N) = \sup_{\beta_1, \dots, \beta_N} \sup_{c \neq 0} \frac{\sum_{k, \ell=1}^N x^{|\beta_k - \beta_\ell|} c_k \bar{c}_\ell}{\sum_{j=1}^N |c_j|^2}$$

is the least upper bound for these eigenvalues, where the suprema are taken over respectively all N -tuples of distinct multi-indices β_1, \dots, β_N and all nonzero vectors $c = (c_1, \dots, c_N)$ in \mathbb{C}^N . We may also refer to $\Lambda_t(N)$ as the supremum of the spectral norms of the matrices $(t^{|\beta_k - \beta_\ell|})$ for fixed N . The problem of estimating $\Lambda_t(N)$ for $t = p^{-\alpha}$ was raised in [7, p. 10].

Trivially, $\Lambda_t(N) \geq \Gamma_t(N)$. In the opposite direction, we have the following estimate.

Theorem 5. *We have*

$$\Lambda_t(N) \leq (e^2 + 1)([\log N] + 2) \max_{1 \leq n \leq N} \Gamma_t(n)$$

whenever $t = (t_j)$ is a decreasing sequence of positive numbers in \mathbb{D}^∞ .

A few remarks are in order before we give the proof of this theorem. First, the result is of interest only when t fails to be in ℓ^1 because if t is in ℓ^1 , then the easy estimate

$$(13) \quad \Lambda_t(N) \leq \prod_{j=1}^{N-1} \frac{1 + t_j}{1 - t_j}$$

which can be obtained from the right-hand side of (12), will be uniformly bounded when $N \rightarrow \infty$. Note that a special version of this estimate is given in [24, p. 152]. We will prove both (13) and a corresponding estimate for the smallest eigenvalue of $(t^{|\beta_k - \beta_\ell|})$ at the end of this section, as a generalization of the result in [24, p. 152].

In our terminology, Dyer and Harman [12] obtained (3) from the estimate

$$\Lambda_{p^{-1/2}}(N) \leq C \exp \left(\frac{c \log N}{\log \log N} \right).$$

Besides the results of [24] and [12], we are not aware of previous estimates of $\Lambda_t(N)$ for any other values of t . If we combine Theorem 1 with Theorem 5, then we obtain precise estimates for $\Lambda_{p^{-\alpha}}(N)$ when $0 < \alpha < 1$. From Gál's theorem (2) and Theorem 5 we also get

$$\Lambda_{p^{-1}}(N) \leq c(\log N)(\log \log N)^2$$

for an absolute constant c , but it is perhaps unlikely this order of growth is optimal. As an application of this result, we note that we may replace λ_N in Theorem 1.1 of [7, p. 10] by our quantity $\Lambda_{p^{-\alpha}}(N)$ and then improve Corollary 1.2 of [7, p. 11] significantly by using our estimates for $\Lambda_{p^{-\alpha}}(N)$.

Let us finally point out that the phenomenon captured by Theorem 4 and Theorem 5 is interesting from a function theoretic point of view: While holomorphic polynomials F of fixed L^2 norm (in terms of their coefficients) are uniformly bounded at any fixed point in $\mathbb{D}^\infty \cap \ell^2$ [10], this is not so in general for the Poisson integrals of $|F|^2$. Indeed, the two theorems give a surprisingly precise statement about the relation between the growth of the number of monomials involved in the polynomials and the growth of such Poisson integrals at points ζ in the complement of $\mathbb{D}^\infty \cap \ell^1$. It is worth noting that the combinatorial Lemma 2 seems indispensable in the proof of these estimates.

Proof of Theorem 5. We will estimate the quadratic form

$$\sum_{k, \ell=1}^N t^{|\beta_k - \beta_\ell|} c_k \bar{c}_\ell$$

for arbitrary multi-indices β_1, \dots, β_N and vectors $c = (c_1, \dots, c_N)$ satisfying $\sum_{j=1}^N |c_j|^2 = 1$. We may clearly assume that the coefficients c_j are nonnegative. Set

$$\mathcal{C}_\ell = \{j : e^{-\ell-1} < c_j \leq e^{-\ell}\}.$$

By the Cauchy–Schwarz inequality, we get

$$(14) \quad \left| \sum_{j=1}^N c_j z^{\beta_j} \right|^2 \leq ([\log N] + 2) \left(\left| \sum_{j: c_j \leq N^{-1}} c_j z^{\beta_j} \right|^2 + \sum_{\ell: 0 \leq \ell < \log N} \left| \sum_{k: k \in \mathcal{C}_\ell} c_k z^{\beta_k} \right|^2 \right).$$

Using (12) and again the Cauchy–Schwarz inequality, we get

$$\int_{\mathbb{T}^K} \left| \sum_{j: c_j \leq N^{-1}} c_j z^{\beta_j} \right|^2 P_K(t, z) d\sigma_K(z) \leq 1.$$

Applying (12) a second time, we also obtain

$$\int_{\mathbb{T}^K} \left| \sum_{k: k \in \mathcal{C}_\ell} c_k z^{\beta_k} \right|^2 P_K(t, z) d\sigma_K(z) \leq e^{-2\ell} (\#\mathcal{C}_\ell) \Gamma_t(\#\mathcal{C}_\ell),$$

which, by the definition of \mathcal{C}_ℓ and the fact that c is a unit vector, implies

$$\sum_{\ell: 0 \leq \ell < \log N} \int_{\mathbb{T}^K} \left| \sum_{k: k \in \mathcal{C}_\ell} c_k z^{\beta_k} \right|^2 P_K(t, z) d\sigma_K(z) \leq e^2 \max_{1 \leq n \leq N} \Gamma_t(n).$$

Returning to (14) and making a final application of (12), we obtain the desired result

$$\Lambda_t(N) \leq ([\log N] + 2)(1 + e^2) \max_{1 \leq n \leq N} \Gamma_t(n).$$

■

Let now $\lambda_t(N)$ denote the infimum of the smallest eigenvalues of the generalized GCD matrices $(t^{|\beta_k - \beta_l|})$ for fixed N . We obtain then the following generalization of the theorem in [24, p. 152].

Theorem 6. *We have*

$$(15) \quad \prod_{j=1}^{N-1} \frac{1 - t_j}{1 + t_j} \leq \lambda_t(N) \leq \Lambda_N(t) \leq \prod_{j=1}^{N-1} \frac{1 + t_j}{1 - t_j}$$

whenever $x = (x_j)$ is a decreasing sequence of positive numbers in \mathbb{D}^∞ .

Proof. Note first that the expressions to the left and to the right are respectively the minimum and the maximum of $P_{N-1}(t, z)$ when z varies over \mathbb{T}^{N-1} . Thus the estimates in (15) follow from (12) if we first make the observation that it suffices to integrate over an $(N-1)$ -circle to compute the $L^2(\sigma_K)$ -norm of a function of the form $\sum_{j=1}^N c_j z^{\beta_j}$. ■

5. PROOF OF THEOREM 2 AND THEOREM 3

Throughout this section we write c for appropriate positive constants, not always the same, which may depend on f , but not on N or anything else. Any additional dependence is explicitly stated, by writing e.g. $c(\varepsilon)$ instead of c . We will use the notation

$$\|g\| = \left(\int_0^1 (g(x))^2 dx \right)^{1/2},$$

where g is assumed to be a real-valued function.

The following lemma is a special case of the Carleson–Hunt inequality [20, Theorem 1].

Lemma 3. *There exists an absolute constant c such that*

$$\int_0^1 \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k \cos 2\pi kx \right| \right)^2 dx \leq c \sum_{k=1}^N c_k^2$$

for any finite sequence $(c_k)_{1 \leq k \leq N}$.

We will use Lemma 3 and Theorem 1 to prove the following two auxiliary results; we will see that Lemma 5 is a fairly easy consequence of Lemma 4.

Lemma 4. *Let $(c_k)_{1 \leq k \leq N}$ be a sequence of real numbers, and let $(n_k)_{1 \leq k \leq N}$ be a strictly increasing sequence of positive integers. Assume that $N^{-2} \leq |c_k| \leq 1$ for $1 \leq k \leq N$. Then for any $\varepsilon > 0$ and for any measurable function f satisfying (4) and either $f \in \text{BV}$ or $f \in \text{Lip}_{1/2}$, we have*

$$\int_0^1 \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k f(n_k x) \right| \right)^2 dx \leq c(\varepsilon) (\log N)^\varepsilon \sum_{k=1}^N c_k^2.$$

Lemma 5. *Let $(c_k)_{k \geq 1}$ be a sequence of real numbers satisfying $|c_k| \leq 1$ for all k . Then for any $\varepsilon > 0$, any measurable function f satisfying (4) and either $f \in \text{BV}$ or $f \in \text{Lip}_{1/2}$, and any $N_1 < N_2$, we have*

$$\left\| \max_{N_1 \leq M \leq N_2} \left| \sum_{k=N_1+1}^M c_k f(kx) \right| \right\| \leq \frac{c}{N_2 - N_1} + c(\varepsilon) (\log(N_2 - N_1))^{\varepsilon/2} \left(\sum_{k=N_1+1}^{N_2} c_k^2 \right)^{1/2}.$$

Proof of Lemma 4. Let f be any function satisfying (4), and assume that either $f \in \text{BV}$ or $f \in \text{Lip}_{1/2}$. To simplify the exposition, we assume that f is even so that its Fourier series is a pure cosine-series:

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx.$$

To make our proof as transparent as possible, we will first prove Lemma 4 when $f \in \text{BV}$. The proof for $f \in \text{Lip}_{1/2}$ is technically more involved and will be given subsequently. In what follows, we will use the notation

$$\delta_i = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof in the case $f \in \text{BV}$: By [28, p. 48], the Fourier coefficients a_j of a function f in BV satisfy

$$(16) \quad |a_j| \leq c j^{-1}, \quad j \geq 1.$$

Set

$$(17) \quad J = \exp((\log N)^{\varepsilon/2})$$

and

$$(18) \quad p(x) = \sum_{j=1}^J a_j \cos 2\pi jx, \quad r(x) = f(x) - p(x).$$

Then, by Minkowski's inequality,

$$(19) \quad \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k f(n_k x) \right| \right\| \leq \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k p(n_k x) \right| \right\| + \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k r(n_k x) \right| \right\|.$$

By (16) and Lemma 3, we have

$$\left\| \max_{1 \leq M \leq N} \left\| \sum_{k=1}^M c_k p(n_k x) \right\| \right\| \leq \sum_{j=1}^J |a_j| \left\| \max_{1 \leq M \leq N} \left\| \sum_{k=1}^M c_k \cos 2\pi j n_k x \right\| \right\| \leq c(\log J) \left(\sum_{k=1}^N c_k^2 \right)^{1/2},$$

which, by the definition of J , gives

$$(20) \quad \left\| \max_{1 \leq M \leq N} \left\| \sum_{k=1}^M c_k p(n_k x) \right\| \right\| \leq c(\log N)^{\varepsilon/2} \left(\sum_{k=1}^N c_k^2 \right)^{1/2}.$$

Estimating the second term on the right-hand side of (19) is more difficult. Let any numbers $0 \leq M_1 < M_2 \leq N$ be given. We want to find a good estimate for

$$(21) \quad \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\|.$$

For any $\ell \in \{0, \lceil 2 \log_2 N \rceil\}$ we define

$$(22) \quad \mathcal{K}_\ell = \{k : M_1 < k \leq M_2 \text{ and } 2^{-\ell-1} < |c_k| \leq 2^{-\ell}\}.$$

Observe that by assumption we have $N^{-2} \leq |c_k| \leq 1$ for $1 \leq k \leq N$. Thus

$$\sum_{\ell=0}^{\lceil 2 \log_2 N \rceil} \sum_{k \in \mathcal{K}_\ell} c_k r(n_k x) = \sum_{k=M_1+1}^{M_2} c_k r(n_k x).$$

Now let any ℓ in $\{0, \lceil 2 \log_2 N \rceil\}$ be fixed, and set $N_\ell = \#\mathcal{K}_\ell$. By (16) and the orthogonality of the trigonometric system, we have

$$(23) \quad \begin{aligned} \int_0^1 \left(\sum_{k \in \mathcal{K}_\ell} c_k r(n_k x) \right)^2 dx &= \frac{1}{2} \sum_{k_1, k_2 \in \mathcal{K}_\ell} \sum_{j_1, j_2=J+1}^{\infty} c_{k_1} c_{k_2} a_{j_1} a_{j_2} \delta_{j_1 k_1 - j_2 k_2} \\ &\leq c 2^{-2\ell} \sum_{k_1, k_2 \in \mathcal{K}_\ell} \sum_{j_1, j_2=J+1}^{\infty} (j_1 j_2)^{-1} \delta_{j_1 k_1 - j_2 k_2}. \end{aligned}$$

Let $v < w$ be two positive integers. Then, following an argument of Koksma [21], we have

$$(24) \quad \begin{aligned} \sum_{j_1, j_2=J+1}^{\infty} (j_1 j_2)^{-1} \delta_{j_1 v - j_2 w} &\leq \sum_{j_1, j_2=1}^{\infty} (j_1 j_2)^{-1} \delta_{j_1 v - j_2 w} \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \frac{\gcd(v, w)}{v} \frac{\gcd(v, w)}{w} \\ &\leq 2 \frac{\gcd(v, w)^2}{vw}. \end{aligned}$$

On the other hand, as in [2, p. 104], we have

$$\begin{aligned}
 \sum_{j_1, j_2=J+1}^{\infty} (j_1 j_2)^{-1} \delta_{j_1 v - j_2 w} &= \sum_{j \geq \lceil (J+1) \gcd(v, w)/v \rceil} \frac{(\gcd(v, w))^2}{j^2 v w} \\
 &\leq \frac{2}{\lceil (J+1) \gcd(v, w)/v \rceil} \frac{(\gcd(v, w))^2}{v w} \\
 &\leq \frac{2 \gcd(v, w)}{J w} \\
 &\leq \frac{2 \gcd(v, w)}{J \sqrt{v w}}.
 \end{aligned}
 \tag{25}$$

Combining (24) and (25), we obtain

$$\begin{aligned}
 \sum_{j_1, j_2=J+1}^{\infty} (j_1 j_2)^{-1} \delta_{j_1 v - j_2 w} &\leq \left(2 \frac{\gcd(v, w)^2}{v w} \right)^{1-\varepsilon} \left(\frac{2 \gcd(v, w)}{J \sqrt{v w}} \right)^{\varepsilon} \\
 &= \frac{2 \gcd(v, w)^{2-\varepsilon}}{J^{\varepsilon} (v w)^{1-\varepsilon/2}}.
 \end{aligned}
 \tag{26}$$

Thus the integral in (23) is bounded by

$$c 2^{-2\ell} \sum_{k_1, k_2 \in K(\ell)} \frac{2 \gcd(n_{k_1}, n_{k_2})^{2-\varepsilon}}{J^{\varepsilon} (n_{k_1} n_{k_2})^{1-\varepsilon/2}},$$

which, by Theorem 1 (for $\alpha = 1 - \varepsilon/2$), is at most

$$c 2^{-2\ell} J^{-\varepsilon} N_{\ell} \exp(c(\varepsilon)(\log N_{\ell})^{\varepsilon/2}(\log \log N_{\ell})^{-1/2}).$$

By Minkowski's inequality and (17), we therefore get the following estimate for (21):

$$\begin{aligned}
 \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\| &\leq \sum_{\ell=0}^{\lceil 2 \log_2 N \rceil} \left\| \sum_{k \in \mathcal{K}_{\ell}} c_k r(n_k x) \right\| \\
 &\leq c \sum_{\ell=0}^{\lceil 2 \log_2 N \rceil} 2^{-\ell} (N_{\ell})^{1/2} J^{-\varepsilon/2} \exp(c(\varepsilon)(\log N_{\ell})^{\varepsilon/2}(\log \log N_{\ell})^{-1/2}).
 \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we infer from this bound that

$$\begin{aligned}
 \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\| &\leq c J^{-\varepsilon/2} (\log N)^{1/2} \left(\sum_{l=0}^{\lceil 2 \log_2 N \rceil} 2^{-2\ell} N_{\ell} \right)^{1/2} \exp(c(\varepsilon)(\log N)^{\varepsilon/2}(\log \log N)^{-1/2}) \\
 &\leq c J^{-\varepsilon/2} (\log N)^{1/2} \left(\sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp(c(\varepsilon)(\log N)^{\varepsilon/2}(\log \log N)^{-1/2})
 \end{aligned}
 \tag{27}$$

$$(28) \quad \leq c(\varepsilon) \exp(-\varepsilon/4 (\log N)^{\varepsilon/2}) \left(\sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2},$$

where we in the last step used the definition of J . Now imitating the proof of the Rademacher-Menshov inequality (see [25, p. 123]), we see that (28) implies

$$(29) \quad \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M c_k r(n_k x) \right| \right\| \leq c(\varepsilon) (\log N)^2 \exp(-\varepsilon/4 (\log N)^{\varepsilon/2}) \left(\sum_{k=1}^N c_k^2 \right)^{1/2} \\ \leq c(\varepsilon) \left(\sum_{k=1}^N c_k^2 \right)^{1/2}.$$

Together with (20), this proves the lemma in the case $f \in \text{BV}$.

Proof in the case $f \in \text{Lip}_{1/2}$: If $f \in \text{Lip}_{1/2}$, then by [28, p. 241] we have

$$(30) \quad \sum_{j=2^{m+1}}^{2^{m+1}} a_j^2 \leq c 2^{-m}, \quad m \geq 0.$$

Note that if $f \in \text{BV}$, then (30) also holds as a consequence of (16); thus the proof for the case $f \in \text{BV}$ could have been included into the present proof. However, (30) is significantly weaker than (16), which makes the proof in the present case more complicated. By the Cauchy-Schwarz inequality, (30) implies that

$$\sum_{j=2^{m+1}}^{2^{m+1}} |a_j| \leq c,$$

and hence

$$(31) \quad \sum_{j=1}^J |a_j| \leq c \log J$$

for any $J \geq 1$. Define J and p, r as in (17) and (18). Then (19) holds, and using (31), we again obtain (20).

We turn to the estimation of the second part in (19). To this end, we set

$$\mathcal{S}_m = \{2^m < j \leq 2^{m+1} : |a_j| \leq 2^{-5m/8}\}, \quad \mathcal{T}_m = \{2^m + 1, \dots, 2^{m+1}\} \setminus \mathcal{S}_m.$$

Then from (30) it is clear that

$$(32) \quad \#\mathcal{T}_m \leq c 2^{m/4}.$$

Note that (30) also implies

$$(33) \quad |a_j| \leq c 2^{-m/2} \quad \text{for all } j \in \{2^m + 1, \dots, 2^{m+1}\}.$$

Let $0 \leq M_1 < M_2 \leq N$ be given, and let μ denote the largest integer such that $2^\mu \leq J$. Two applications of Minkowski's inequality give

$$\begin{aligned} \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\| &\leq \sum_{m=\mu}^{\infty} \left\| \sum_{k=M_1+1}^{M_2} \sum_{j=2^{m+1}}^{2^{m+1}} |a_j| |c_k| \cos 2\pi j n_k x \right\| \\ &\leq \sum_{m=\mu}^{\infty} \left(\left\| \sum_{k=M_1+1}^{M_2} \sum_{j \in \mathcal{S}_m} |a_j| |c_k| \cos 2\pi j n_k x \right\| + \left\| \sum_{k=M_1+1}^{M_2} \sum_{j \in \mathcal{T}_m} |a_j| |c_k| \cos 2\pi j n_k x \right\| \right). \end{aligned}$$

We reverse the order of summation and use Minkowski's inequality along with (33), (32), and the orthogonality of the trigonometric system to estimate the second norm on the right-hand side of this inequality. Using also the definition of \mathcal{S}_m to deal with the first norm, we therefore get:

$$(34) \quad \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\| \leq \sum_{m=\mu}^{\infty} \left(\left\| \sum_{k=M_1+1}^{M_2} \sum_{j \in \mathcal{S}_m} j^{-5/8} |c_k| \cos 2\pi j n_k x \right\| + c 2^{-m/4} \left(\sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \right).$$

Now let m be fixed. We define \mathcal{K}_ℓ as in (22), and observe that

$$(35) \quad \int_0^1 \left(\sum_{k \in \mathcal{K}_\ell} \sum_{j \in \mathcal{S}_m} j^{-5/8} |c_k| \cos 2\pi j n_k x \right)^2 dx \leq c 2^{-2\ell} \sum_{k_1, k_2 \in \mathcal{K}_\ell} \sum_{j_1, j_2=2^m+1}^{\infty} (j_1 j_2)^{-5/8} \delta_{j_1 k_1 - j_2 k_2}.$$

Instead of (24), we get

$$(36) \quad \begin{aligned} \sum_{j_1, j_2=2^m+1}^{\infty} (j_1 j_2)^{-5/8} \delta_{j_1 v - j_2 w} &= \sum_{j=1}^{\infty} \frac{1}{j^{10/8}} \frac{\gcd(v, w)}{v} \frac{\gcd(v, w)}{w} \\ &\leq c \frac{\gcd(v, w)^2}{vw}, \end{aligned}$$

and as a replacement for (25), we have

$$(37) \quad \begin{aligned} \sum_{j_1, j_2=2^m+1}^{\infty} (j_1 j_2)^{-5/8} \delta_{j_1 v - j_2 w} &= \sum_{j \geq \lceil (2^m+1) \gcd(v, w)/v \rceil} \frac{(\gcd(v, w))^2}{j^{-10/8} vw} \\ &\leq \frac{c}{2^{m/4}} \frac{(\gcd(v, w))^{7/4}}{(vw)^{7/8}}. \end{aligned}$$

Combining (36) and (37) we have

$$\begin{aligned} \sum_{j_1, j_2=2^m+1}^{\infty} (j_1 j_2)^{-5/8} \delta_{j_1 v - j_2 w} &\leq c \left(\frac{\gcd(v, w)^2}{vw} \right)^{1-4\epsilon} \left(\frac{1}{2^{m/4}} \frac{(\gcd(v, w))^{7/4}}{(vw)^{7/8}} \right)^{4\epsilon} \\ &\leq c 2^{-m\epsilon} \left(\frac{\gcd(v, w)^2}{vw} \right)^{1-\epsilon}, \end{aligned}$$

and consequently the expression in (35) is at most

$$c2^{-2\ell} \sum_{k_1, k_2 \in K(\ell)} 2^{-m\varepsilon} \frac{(\gcd(n_{k_1}, n_{k_2}))^{2-\varepsilon}}{(n_{k_1} n_{k_2})^{1-\varepsilon/2}},$$

and, as in (27), we obtain the upper bound

$$\begin{aligned} & \left\| \sum_{k=M_1+1}^{M_2} \sum_{j \in \mathcal{S}_m} j^{-5/8} |c_k| \cos 2\pi j n_k x \right\| \\ (38) \quad & \leq c2^{-m\varepsilon/2} (\log N)^{1/2} \left(\sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp(c(\varepsilon)(\log N)^{\varepsilon/2} (\log \log N)^{-1/2}). \end{aligned}$$

Together with (34) this yields

$$\begin{aligned} & \left\| \sum_{k=M_1+1}^{M_2} c_k r(n_k x) \right\| \\ & \leq c(\varepsilon) J^{-\varepsilon/2} (\log N)^{1/2} \left(\sum_{k=M_1+1}^{M_2} c_k^2 \right)^{1/2} \exp(c(\varepsilon)(\log N)^{\varepsilon/2} (\log \log N)^{-1/2}). \end{aligned}$$

The rest of the proof can be carried out as in the case when $f \in \text{BV}$. ■

Proof of Lemma 5. To obtain Lemma 5 from Lemma 4, set $N = N_2 - N_1$, and define for $1 \leq k \leq N$

$$\tilde{c}_k = \begin{cases} 0 & \text{if } |c_{N_1+k}| \leq N^{-2} \\ c_{N_1+k} & \text{otherwise} \end{cases}$$

and

$$d_k = \begin{cases} c_{N_1+k} & \text{if } |c_{N_1+k}| \leq N^{-2} \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any $k \in \{1, \dots, N\}$ we have $c_{N_1+k} = \tilde{c}_k + d_k$, and consequently by Minkowski's inequality

$$\begin{aligned} & \left\| \max_{N_1 \leq M \leq N_2} \left| \sum_{k=N_1+1}^M c_k f(kx) \right| \right\| \\ (39) \quad & \leq \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M \tilde{c}_k f((N_1 + k)x) \right| \right\| + \left\| \max_{1 \leq M \leq N} \left| \sum_{k=1}^M d_k f((N_1 + k)x) \right| \right\|. \end{aligned}$$

Furthermore, we clearly have $|d_k| \leq N^{-2}$ and $|\tilde{c}_k| \geq N^{-2}$ for all k . Using Lemma 4 for $n_k = N_1 + k$, $1 \leq k \leq N$, and for the sequence $(\tilde{c}_k)_{1 \leq k \leq N}$, we get

$$(40) \quad \int_0^1 \left(\max_{1 < M \leq N} \left| \sum_{k=1}^M \tilde{c}_k f((N_1 + k)x) \right| \right)^2 dx \leq c(\varepsilon) (\log N)^\varepsilon \sum_{k=N_1+1}^{N_2} c_k^2.$$

We also have, for $1 \leq M \leq N$,

$$(41) \quad \left| \sum_{k=1}^M d_k f((N_1 + k)x) \right| \leq c \frac{M}{N^2} \leq \frac{c}{N}.$$

Combining (39), (40), and (41), we arrive at the conclusion of the lemma. ■

Proof of Theorem 2. Let the sequence $(n_k)_{k \geq 1}$ and any $\varepsilon > 0$ be given. For any $m \geq 1$, define

$$A_m = \left\{ x \in (0, 1) : \max_{2^m < M \leq 2^{m+1}} \left| \sum_{k=1}^M f(n_k x) \right| \geq \sqrt{2^{m+1}} (\log(2^{m+1}))^{1/2+\varepsilon} \right\}.$$

We view Lebesgue measure on $(0, 1)$ as a probability measure and let $|A|$ denote the measure of a subset A of $(0, 1)$. Using Lemma 4 we obtain

$$\left\| \max_{2^m < M \leq 2^{m+1}} \left| \sum_{k=1}^M f(n_k x) \right| \right\| \leq c(\varepsilon) (\log(2^{m+1}))^{\varepsilon/2} \sqrt{2^{m+1}}.$$

Thus the Markov inequality implies that

$$|A_m| \leq \frac{c(\varepsilon)}{(\log(2^{m+1}))^{1+\varepsilon}} \leq c(\varepsilon) m^{-1-\varepsilon}.$$

We may therefore infer from the Borel–Cantelli lemma that, with probability 1, only finitely many events A_m occur. Consequently,

$$\left| \sum_{k=1}^N f(n_k x) \right| = \mathcal{O} \left(\sqrt{N} (\log N)^{1/2+\varepsilon} \right)$$

for almost all $x \in (0, 1)$. ■

Proof of Theorem 3. Let $(c_k)_{k \geq 1}$ be a sequence of real numbers such that for some fixed $\mu > 0$ we have

$$(42) \quad \sum_{k=1}^{\infty} c_k^2 (\log k)^\mu < \infty.$$

Without loss of generality, we may assume that $|c_k| \leq 1$, $k \geq 1$. If we can show that for any given $\eta > 0$ there exists an $M_0(\eta)$, such that

$$(43) \quad \left\| \sup_{M > M_0} \left| \sum_{k=M_0+1}^M c_k f(kx) \right| \right\| \leq \eta,$$

then, by [1, Lemma 6], this implies the a.e. convergence of $\sum_k c_k f(kx)$. For $h \geq 1$ define

$$N(h) = \max\{k : k \leq \exp(h^{4/\mu})\}.$$

By (42), we have

$$(44) \quad \sum_{k=N(h)+1}^{N(h+1)} c_k^2 \leq c(\log N(h))^{-\mu} \leq ch^{-4}.$$

For any $H \geq 1$, by (44), the monotone convergence theorem, Minkowski's inequality, and using Lemma 5 for $\varepsilon = \mu/2$, we have

$$(45) \quad \begin{aligned} \left\| \sup_{M \geq N(H)} \left| \sum_{k=N(H)+1}^M c_k f(kx) \right| \right\| &\leq \sum_{h=H}^{\infty} \left\| \max_{N(h) < M \leq N(h+1)} \left| \sum_{k=N(h)+1}^M c_k f(kx) \right| \right\| \\ &\leq \sum_{h=H}^{\infty} (c(N(h+1) - N(h))^{-1} + c(\varepsilon)(\log(N(h+1)))^{\varepsilon} h^{-4}) \\ &\leq \sum_{h=H}^{\infty} c(\varepsilon) h^{-2}. \end{aligned}$$

Now it is clear that choosing $H = H(\eta)$ sufficiently large, we can make the expression in (45) arbitrarily small, which means that we can obtain (43) for any given $\eta > 0$. \blacksquare

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